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Boundary Conditions for Time Dependent Problems with an Artificial Boundary

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In this paper different procedures for treating artificial boundaries are analyzed, and it is shown that many commonly used methods give bad results. A new procedure is developed for the case where the asymptotic behaviour of the coefficient matrices is known.

1. INTRODUCTION

Many physical problems require the solution of partial differential equations on some infinite domain Ω with boundary $\partial\Omega$. An example is given in fig. 1. For computational reasons Ω is replaced by a finite domain Ω_1 and the problem arises to specify boundary conditions at the artificial boundary B_2. Consider, for example, the differential equations for a nonviscous fluid which at subsonic speed leaves Ω_1 through the

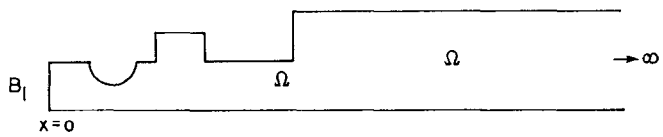

FIGURE 1

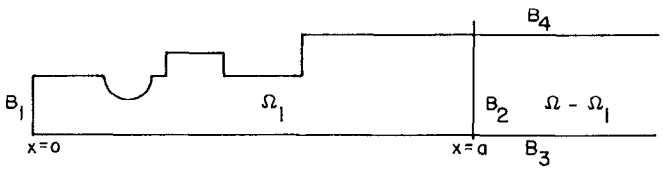

FIGURE 2

boundary B_2. Then there is one characteristic which points back into the region Ω_1 and therefore one boundary condition has to be given. In general, no detailed knowledge of the solution on B_2 is given and other principles have to be applied. For example, if one solves the problem by a difference approximation then one often uses

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upstream differencing on B_2 for all the dependent variables (see for example Roache [4]). This procedure is sometimes combined with overspecifying the solution on the boundary B_1.

Recently, B. Engquist and A. Majda [1] have proposed another principle, namely, to specify the boundary conditions on B_2 in such a way that no reflection takes place.

In this paper we want to investigate when these principles garantuee that the solution of the simplified problem is close to the solution of the original one. A necessary condition for this is that in $\Omega - \Omega_1$ the solution of the original problem only changes slowly with respect to space and time. Therefore we can linearize the problem and we assume that the linearized equations represent a hyperbolic first order system. This is true for ideal flow problems.

In the next section we consider the model equation

$$\begin{aligned} \partial u/\partial t &= \partial u/\partial x, \qquad x \geqslant 0, \quad t \geqslant 0, \\ u(x, 0) &= f(x) \qquad \text{for} \quad t = 0, \end{aligned} \tag{1.1}$$

on the half line $0 \leqslant x < \infty$ and approximate it by

$$\begin{aligned} \partial v/\partial t &= \partial v/\partial x, \qquad 0 \leqslant x \leqslant a, \quad t \geqslant 0, \\ v(x, 0) &= f(x) \qquad \text{for} \quad t = 0, \end{aligned} \tag{1.2}$$

on the finite interval $0 \leqslant x \leqslant a$. The last problem is well posed if we specify boundary conditions at $x = a$ (but not at $x = 0$).

We solve (1.2) by the Lax-Wendroff difference scheme. As boundary conditions for the difference approximation we have a number of different possibilities.

a) For $x = 0$ we either specify v, for example,

$$v(0, t) = 0,$$

or we use an extrapolation procedure

$$(hD_+)^p v(0, t) = 0, \; hD_+ v(0, t) = v(h, t) - v(0, t).$$

For $p = 2$ this is the usual upstream differencing.

b) For $x = a$ we have the analogous possibilities.

We obtain the following results.

1. For $x = 0$ one shall not overspecify, i.e. v shall not be given. Upstream differencing gives good results.

2. If $f(x) \approx$ const. for $x \geqslant a$ then

$$hD_- v(a, t) = v(a, t) - v(a - h, t) = 0 \tag{1.3}$$

can be used. Upstream differencing at $x = a$ can give completely wrong results.

3. The principle of no reflection at $x = a$ is useful only if $f(x) \approx$ const. for $x \geqslant a$.

In section 3 we show that the corresponding conclusions hold for systems

$$\partial v/\partial t = A\, \partial v/\partial x$$

where A is a constant matrix.

In section 4 we make a thorough investigation of systems with variable coefficients

$$\partial u/\partial t = A(x, t)\, \partial u/\partial x.$$

We show that extrapolation procedures or the principle of no reflection is useful only if $A(x, t)$ is essentially independent of x for $x \geqslant a$, or that the solution is highly oscillatory in time.

If one knows the asymptotic behavior of $A(x, t)$, for example,

$$A(x, t) = A_0 + x^{-2}A_1(x, t)$$

then one can derive new principles, which are useful for steady state calculations.

The last section is concerned with system

$$\partial u/\partial t = A\, \partial u/\partial x + B\, \partial u/\partial y$$

in two space dimensions. The above principles are useful if the influence of $B\, \partial u/\partial y$ is small, i.e. the problem is essentially one dimensional. For cases where this does not hold we again construct new principles.

In many applications the time dependent equations are used to obtain the steady state solution. We shall also study the effect of our boundary conditions on the convergence rate to the steade state.

2. The Model Problem

We want to solve the differential equation (1.2) by the Lax-Wendroff scheme. Let $k > 0$, $h = a/N$, N natural number, denote the time step and the space step respectively. The gridpoints are given by $x_\nu = \nu h$, $t_n = nk$, $\nu = 0, 1, \ldots, N$; $n = 0, 1, 2, \ldots$; and the gridfuntions by $w_\nu^n = w(x_\nu, t_n)$. We approximate (1.2) by

$$\begin{aligned} w_\nu^{n+1} &= \left(I + kD_0 + \frac{k^2}{2} D_+D_-\right) w_\nu^n, \qquad 0 < \nu < N, \\ w_\nu^0 &= f_\nu, \end{aligned} \tag{2.1}$$

where

$$2hD_0w_\nu = w_{\nu+1} - w_{\nu-1}, \; hD_+w_\nu = w_{\nu+1} - w_\nu, \; hD_-w_\nu = w_\nu - w_{\nu-1},$$

denote the usual centered, forward and backward difference operators respectively. We assume that $k/h < 1$, which grantuees that the approximation is dissipative.

We first study the case that proper boundary conditions

$$(hD_+)^p w_0^n = 0,\ p \geqslant 1,\ w_N^n = g \tag{2.2}$$

are given. Then the approximation (2.1), (2.2) is stable (see [2]) and in every finite time interval the solution of (2.1), (2.2) converges for $h \to 0$ to the corresponding solution of (1.2) which satisfies the boundary condition

$$v(a, t) = g. \tag{2.3}$$

We now study the behaviour of (2.1), (2.2) for $t \to \infty$ with fixed k, h. We have

THEOREM 2.1. *Let k, h be fixed. The solutions of* (2.1), (2.2) *converge exponentially fast to the steady state solution*

$$w_\nu \equiv g, \qquad \nu = 0, 1, \ldots, N \tag{2.4}$$

as $n \to \infty$.

Proof. We write (2.1), (2.2) in matrix form

$$\bar{w}^{n+1} = Q\bar{w}^n + \bar{b},\ \bar{b} = (0, 0, \ldots, 0, g)' \tag{2.5}$$

where $\bar{w} = (w_0, \ldots, w_N)'$ is a column vector and Q an $(N+1) \times (N+1)$ matrix. It is well known that the solutions of (2.1), (2.2) converge to the steady state solution (2.4) if and only if the eigenvalues z of Q satisfy $|z| < 1$. The eigensolutions $\bar{\phi}$ have the form

$$\phi_\nu = \sigma_1 \kappa_1^\nu + \sigma_2 \kappa_2^\nu, \qquad \nu = 0, 1, 2, \ldots, N, \qquad \kappa_1 \neq \kappa_2$$

where κ_1, κ_2 are the roots of

$$z\kappa = \kappa + \tfrac{1}{2}\lambda(\kappa^2 - 1) + \tfrac{1}{2}\lambda^2(\kappa - 1)^2 \tag{2.6}$$

$\bar{\phi}$ must satisfy the homogeneous boundary conditions (2.2), i.e.

$$\sigma_1(\kappa_1 - 1)^p + \sigma_2(\kappa_2 - 1)^p = 0,\ \sigma_1\kappa_1^N + \sigma_2\kappa_2^N = 0.$$

This system has a nontrivial solution if and only if

$$(\kappa_1 - 1)^p \kappa_2^N = \kappa_1^N(\kappa_2 - 1)^p. \tag{2.7}$$

An easy calculation shows that for sufficiently small h the relations (2.6) (2.7) imply

$$|z| < 1 - \delta, \delta > 0 \qquad \text{independent of } h$$

for all eigenvalues of Q. This proves the theorem.

We next investigate the case where we use extrapolation also at $x = a$, i.e. we replace (2.2) by

$$hD_+w_0^n = hD_+w_{N-1}^n = 0. \qquad \text{(For simplicity we set } p = 1.\text{)} \tag{2.8}$$

In this case we have

THEOREM 2.2. *Assume that $v(x, 0) = f(x)$ is a smooth function with $df(a)/dx = 0$. Then the solutions of* (2.1), (2.8) *converge for $h \to 0$ to the solution of* (1.2) *which satisfies the boundary condition*

$$v(a, t) = f(a) \tag{2.9}$$

Proof. Let

$$\tilde{w}_\nu^n = D_+w_\nu^n, \qquad \nu = 0, 1, 2, \ldots, N - 1.$$

Then $\tilde{w}_\nu^n$ is the solution of (2.1) which satisfies the initial and boundary conditions

$$\tilde{w}_\nu^0 = D_+f_\nu, \quad \nu = 1, 2, \ldots, N - 2; \qquad \tilde{w}_0^n = \tilde{w}_{N-1}^n = 0, \quad n = 0, 1, 2, \ldots .$$

S. Parter [3] has shown that the $\tilde{w}_\nu^n$ are uniformly bounded and for any $\delta > 0$

$$\sup_{\delta \leqslant x_\nu \leqslant a, 0 \leqslant t_n \leqslant T} |y(x_\nu, t_n) - \tilde{w}_\nu^n| \to 0 \qquad \text{as} \quad h \to 0. \tag{2.10}$$

Here y is the solution of (1.2) with initial and boundary conditions

$$y(x, 0) = df(x)/dx, \; y(a, t) = 0 = df(a)/dx.$$

Therefore we have also that

$$\lim_{h \to 0} D_0w_{N-1}^n = y(a, t_n) = 0, \qquad \lim_{h \to 0} hD_+D_-w_{N-1}^n = 0$$

and by (2.1)

$$\lim_{h \to 0} \frac{w_{N-1}^{n+1} - w_{N-1}^n}{k} = 0, \qquad \text{i.e.} \quad \lim_{h \to 0} w_{N-1}^n = f(a).$$

Now

$$w_j^n = -\sum_{\nu=j}^{N-2} D_+w_\nu^n h - w_{N-1}^n = -\sum_{\nu=j}^{N-2} \tilde{w}^n h - w_{N-1}^n$$

and by (2.10)

$$\sup_{\delta \leqslant x_\nu \leqslant a, 0 \leqslant t_n \leqslant T} | v(x_\nu, t) - w(x_\nu, t)| \to 0 \qquad \text{as} \quad h \to 0,$$
$$v(x, t) = f(a) - \int_x^a y(\xi, t)\, d\xi. \tag{2.11}$$

$v(x, t)$ is the solution of (2.1) which satisfies the boundary condition (2.9). Also, for $0 \leqslant x_\nu \leqslant \delta$ we have

$$\begin{aligned} | v(x_\nu, t) - w(x_\nu, t)| &\leqslant | v(\tilde{x}, t) - w(\tilde{x}, t)| + | v(x_\nu, t) - v(\tilde{x}, t)| \\ &\quad + | w(\tilde{x}, t) - w(x_\nu, t)| \leqslant \text{const. } \delta, \end{aligned} \tag{2.12}$$

where $\tilde{x} = x_\mu$ is a gridpoint with $\delta \leqslant x_\mu \leqslant \delta + h$. This proves the theorem.

For steady state calculations we have

THEOREM 2.3. *For fixed k, h the scheme* (2.1), (2.8) *is weakly convergent as $n \to \infty$ i.e. the limit function w^∞ depends on the initial values.*

Proof. We write (2.1), (2.8) again in the matrix form (2.5). An easy calculation shows that $z = 1$ is a simple eigenvalue corresponding to the eigenfunction $\vec{\phi} = \text{const.}$. All the remaining eigenvalues satisfy again $| z | \leqslant 1 - \delta < 1$. This proves the theorem.

Thus the steady state depends on the initial function $f(x)$. If we replace $f(x)$ by $f(x) + \text{const.}$, then also the steady state changes by this constant. Observe that the rate of convergence is the same as in theorem 2.1.

As a third alternative we consider

$$w_0^n = g, \qquad hD_+ w_{N-1}^n = 0, \tag{2.13}$$

i.e. the boundary values are given on the "wrong side". This corresponds, for example, to the subsonic case in fluid mechanics where all the variables are prescribed on the inflow side. In a similar way as theorem 2.2 on can prove

THEOREM 2.4. *Assume that the conditions of theorem* 2.2 *are satisfied. Then the solution of* (2.1), (2.13) *converges for $h \to 0$ on any finite domain $\delta \leqslant x \leqslant a$, $0 \leqslant t \leqslant T$ to the solution of* (1.2), (2.9).

For steady state calculations we have

THEOREM 2.5. *For fixed k, h the solution of* (2.1), (2.13) *converges for $n \to \infty$ to the unique steady state $w_\nu^\infty \equiv g$, $\nu = 0, 1, 2, \ldots, N$ if and only if N is odd. Also, the speed of convergence is extremely slow.*

Proof. The condition (2.7) for the existence of a nontrivial solution becomes now

$$(\kappa_1 - 1)\, \kappa_1^{N-1} = \kappa_2^{N-1}(\kappa_2 - 1). \tag{2.14}$$

Using (2.6) and (2.14) an easy calculation shows that there is one eigenvalue of the form

$$z \approx 1 + (-1)^N \lambda \frac{2-\lambda}{1+\lambda} \left(\frac{1-\lambda}{1+\lambda}\right)^{N-1}, \qquad \lambda = k/h,$$

while all the other eigenvalues satisfy $|z| \leqslant 1 - \delta < 1$. Therefore, we obtain growing solutions if N is even, but converging solutions if N is odd. The rate of convergence in the last case is determined by the above eigenvalue, i.e. it is extremely slow. Furthermore, the solution will contain oscillatory components near the boundary.

For converging solutions the steady state is given by

$$w_\nu^\infty = \sigma_1 \kappa_1^\nu + \sigma_2 \kappa_2^\nu \tag{2.15}$$

where κ_1, κ_2 are the roots of

$$\kappa^2 - 1 + \lambda(\kappa - 1)^2 = 0, \text{ i.e. } \kappa_1 = (1-\lambda)/(1+\lambda), \ \kappa_2 = 1$$

w_ν^∞ must satisfy the boundary conditions (2.13). Therefore

$$\sigma_1 + \sigma_2 = g, \qquad \sigma_1 \kappa_1^{N-1}(\kappa_1 - 1) + \sigma_2 \kappa_2^{N-1}(\kappa_2 - 1) = 0$$

which gives us $\sigma_1 = 0$, $\sigma_2 = g$ and the theorem follows.

By theorem 2.4 the solution of (2.1). (2.13) converges to the solution of (1.2), (2.9) on every finite time interval. Since $v(x, t) \equiv f(a)$ for $t \geqslant a$ this seems to indicate a contradiction to the result of theorem 2.5. However, it can be explained in the following way. For fixed values of k, h, a typical calculation will after a relatively short time give a good approximation of $v(x, t) \equiv f(a)$, but it will later change very slowly and finally converge to g. The following calculation illustrates this point. We compute the solution of (2.1), (2.13) on the interval $0 \leqslant x \leqslant 1$ using

$$w_\nu^0 = f_\nu = \sin x_\nu, \qquad w_0^n = hD_+ w_{N-1}^n = 0$$

as initial and boundary conditions. The figures show $w(2/3, t)$ as a function of time in two different scales. It is very tempting to consider the solution as a steady state after a relatively short time. However, as can be seen from fig. 3 and fig. 4, the solution at $t = 2$ has nothing to do with the final steady state.

We can also use higher order extrapolation at the boundary

$$(hD_+)^p w_0^n = (hD_+)^p w_{N-p}^n = 0, \qquad p > 1. \tag{2.16}$$

The condition (2.7) for a nontrivial solution becomes

$$(\kappa_1 - 1)^p (\kappa_2 - 1)^p (\kappa_2^{N-p} - \kappa_1^{N-p}) = 0$$

and there is an eigenvalue $z = 1$ of order p which has only one eigensolution. There-

fore there are solutions which grow like t^{p-1} and for steady state calculations no convergence will occur.

For calculations on a finite time interval, we can, using the same technique as in theorem 2.2, prove that the solutions converge for $h \to 0$ to a solution of (1.2) which satisfies the boundary condition

$$\partial^p v(a, t)/\partial x^p = 0.$$

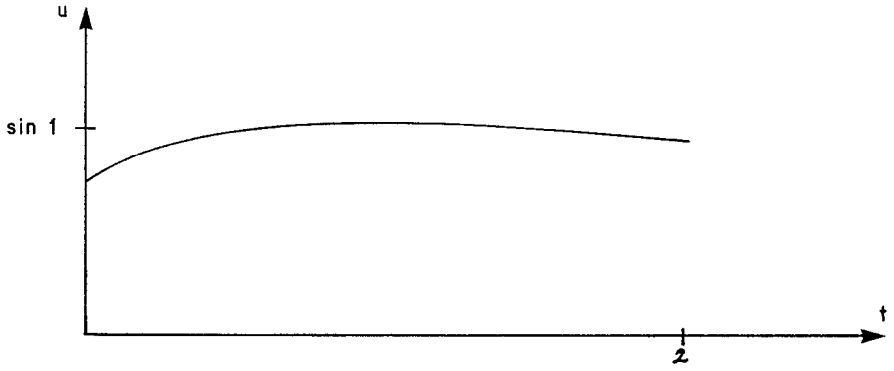

FIGURE 3

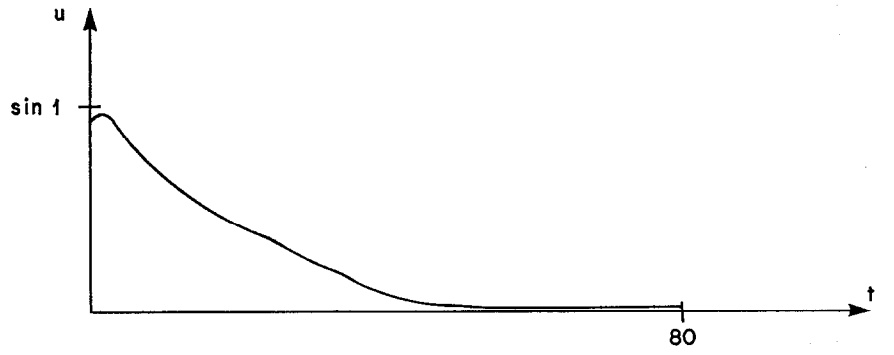

FIGURE 4

The results in this section can also be generalized to equations

$$\partial v/\partial t = \alpha\, \partial v/\partial x, \qquad \alpha > 0$$

and difference schemes

$$w_\nu^{n+1} = (I + \alpha k D_0 + \beta k^2 D_+ D_-)\, w_\nu^n.$$

Stability for the Cauchy problem requires

$$\beta \leqslant \tfrac{1}{2},\ \lambda^2 \leqslant 2\beta \qquad \text{where} \quad \lambda = \alpha k/h.$$

We note in particular, that for the case of overspecification at $x = 0$, we get convergence to a steady state if and only if at least one of the following conditions is satisfied.

1. N is odd, 2. $\lambda \leqslant 2\beta$.

Therefore, by strengthening the von Neumann condition, we can avoid the restriction on N. However, the speed of convergence is still extremely low.

3. Difference Approximations for Systems with Constant Coefficients

In this section we will make a few comments on the numerical solution of systems

$$\partial \tilde{v}/\partial t = A\, \partial \tilde{v}/\partial x, \qquad 0 \leqslant x \leqslant a$$

where $\tilde{v} = (\tilde{v}^{(1)},\ldots, \tilde{v}^{(n)})'$ is a vector function with n components and A a constant $n \times n$ matrix. We assume that A can be transformed to diagonal form and that the eigenvalues of A are real and nonzero, i.e. there exist a nonsingular transformation S such that

$$\Lambda = S^{-1}AS = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \tag{3.1}$$

where

$$\Lambda_1 = \begin{pmatrix} \lambda_1 & 0 \cdots 0 \\ 0 & \lambda_2 \; 0 \cdots 0 \\ \cdots\cdots\cdots \\ 0 \cdots 0 & \lambda_p \end{pmatrix} > 0, \qquad \Lambda_2 = \begin{pmatrix} \lambda_{r+1} & 0 \cdots 0 \\ 0 & \lambda_{r+2} \; 0 \cdots 0 \\ \cdots\cdots\cdots \\ 0 \cdots 0 & \lambda_n \end{pmatrix} < 0 \tag{3.2}$$

are positive and negative definite diagonal matrices respectively.

The Lax-Wendroff scheme is given by

$$\tilde{w}_\nu^{n+1} = (I + kAD_0 + \tfrac{1}{2}k^2A^2D_+D_-)\, \tilde{w}_\nu^n, \qquad 0 < \nu < N. \tag{3.3}$$

To determine its solution we have again to specify boundary conditions at $x = 0$ and $x = a$. If the boundary conditions are of the same type for all components of $\tilde{w}$, for example

$$\tilde{w}_0^{\,n} = \tilde{g}, \qquad hD_+\tilde{w}_{N-1}^n = 0, \tag{3.4}$$

then all the results of the last section apply. We need only to introduce new variables $w_\nu = S^{-1}\tilde{w}_\nu$ to obtain

$$\begin{gathered} w_\nu^{n+1} = (I + k\Lambda D_0 + \tfrac{1}{2}k^2\Lambda^2 D_+D_-)\, w_\nu^n, \qquad 0 < \nu < N \\ w_0^{\,n} = g, \qquad hD_+w_{N-1}^n = 0 \end{gathered} \tag{3.5}$$

i.e. n decoupled problems of type (2.1), (2.13).

This transformation can also be used to discuss general boundary fconditions. For example, specify at $x = 0$ correct boundary conditions and extrapolate at $x = a$ all variables. Then, after transformation, these boundary conditions become

$$(w_0^{\mathrm{II}})^n = R_0(w_0^{\mathrm{I}})^n + (g_0^{\mathrm{II}})^n, \qquad (hD_+)^p\,(w_0^{\mathrm{I}})^n = Q(hD_+)^p\,(w_0^{\mathrm{II}})^n, \qquad hD_+w_{N-1}^n = 0. \tag{3.6}$$

Here w^{I}, w^{II} correspond to the partition of Λ and R, Q denote $(n - r) \times r$ and $r \times (n - r)$ matrices respectively. Then w converges to a solution of

$$\partial v/\partial t = \Lambda\, \partial v/\partial x \tag{3.7}$$

with boundary conditions

$$v^{\mathrm{II}}(0, t) = R_0v^{\mathrm{I}}(0, t) + g_0^{\mathrm{II}}(t), \qquad \partial v^{\mathrm{I}}(a, t)/dx = 0. \tag{3.8}$$

By (3.7) the relation $\partial v^{\mathrm{I}}/\partial x = 0$ implies $\partial v^{\mathrm{I}}/\partial t = 0$ and therefore

$$v^{\mathrm{I}}(a, t) \equiv v^{\mathrm{I}}(a, 0). \tag{3.9}$$

Thus in steady state calculations the steady state depends on the initial values.

4. General Considerations for Problems in one Space Dimension

In this section we consider hyperbolic systems

$$\partial\tilde{u}/\partial t = A(x, t)\,\partial\tilde{u}/\partial x$$

with variable coefficients in the quarter space $0 \leqslant x < \infty$, $t \geqslant 0$ where the matrix A depends smoothly on x, t. We assume that there is a smooth transformation $S(x, t)$ such that (3.1), (3.2) hold for every fixed x, t. Then we can introduce new dependent variables $u = S^{-1}\tilde{u}$ and obtain

$$\partial u/\partial t = \Lambda\, \partial u/\partial x + Bu, \; B = -S^{-1}\,\partial S/\partial t + S^{-1}A\,\partial S/\partial x. \tag{4.1}$$

For $t = 0$ initial values

$$u(x, 0) = f(x), \qquad 0 \leqslant x < \infty \tag{4.2}$$

and for $x = 0$ boundary conditions

$$u^{\mathrm{II}}(0, t) = R_0u^{\mathrm{I}}(0, t) + g_0^{\mathrm{II}}(t), \qquad t \geqslant 0 \tag{4.3}$$

are given. Here $f(x)$, $g_0^{\mathrm{II}}(t)$ are smooth functions, $u^{\mathrm{I}} = (u^{(1)},\ldots, u^{(r)})'$, $u^{\mathrm{II}} = (u^{(r+1)},\ldots, u^{(n)})'$ correspond to the partition of Λ, and R_0 is an $(n - r) \times r$ matrix. Thus the

number of boundary conditions is equal to the number of characteristics which enter the region $x \geqslant 0$, $t \geqslant 0$.

If we want to solve the above problem numerically we have to replace the infinite interval $0 \leqslant x < \infty$ by a finite one. There are two ways to do this.

1. Determine some transformation of the independent variable x which transforms $0 \leqslant x < \infty$ into $0 \leqslant x \leqslant a$.

2. Replace $0 \leqslant x < \infty$ directly by $0 \leqslant x \leqslant a$ and find at $x = a$ a function $g_1^{\mathrm{I}}(t)$ such that the solution of

$$\partial v/\partial t = \Lambda\, \partial v/\partial x + Bv, \qquad 0 \leqslant x \leqslant a, \quad t \geqslant 0 \tag{4.4}$$

with initial conditions

$$v(x, 0) = f(x), \qquad 0 \leqslant x \leqslant a \tag{4.5}$$

and boundary conditions

$$v^{\mathrm{II}}(0, t) = R_0 v^{\mathrm{I}}(0, t) + g_0^{\mathrm{I}}(t), \qquad v^{\mathrm{I}}(a, t) = g_1^{\mathrm{I}}(t), \quad t \geqslant 0 \tag{4.6}$$

differs at most slightly from the solution of the original problem (4.1)–(4.3).

Connected with the above equations is the following problem. Consider the system (4.1)

$$\partial y/\partial t = \Lambda\, \partial y/\partial x + By, \qquad x \geqslant a, \qquad t \geqslant 0 \tag{4.7}$$

for $x \geqslant a$ with initial values

$$y(x, 0) = f(x), \qquad x \geqslant a \tag{4.8}$$

and boundary conditions

$$y^{\mathrm{II}}(a, t) = g_1^{\mathrm{II}}(t). \tag{4.9}$$

We can express the solution $u(x, t)$ of (4.1)–(4.3) with help of $v(x, t)$ and $y(x, t)$. The following lemma is obvious.

LEMMA 4.1.

$$u(x, t) = \begin{cases} v(x, t), & x \leqslant a \\ y(x, t), & x \geqslant a \end{cases}, \quad t \geqslant 0, \tag{4.10}$$

if and only if

$$v^{\mathrm{I}}(a, t) = g_1^{\mathrm{I}}(t) = y^{\mathrm{I}}(a, t), \qquad v^{\mathrm{II}}(a, t) = g_1^{\mathrm{II}}(t) = y^{\mathrm{II}}(a, t). \tag{4.11}$$

We can state this result in another way. Let $f(x)$ be fixed. Then the problem (4.7)–

(4.9) has for every $g_1^{II}(t)$ a unique solution $y(x, t)$. In particular, $y^I(a, t)$ is uniquely determined by $y^{II}(a, t) = g_1^{II}(t)$. Thus there is a linear operator $R_1(t)$ such that

$$y^I(a, t) = R_1(t)\, y^{II}(a, t) + b^I(t). \tag{4.12}$$

Here $b^I(t)$ is determined by $f(x)$ and $R_1(t)$ depends on $y^{II}(a, \xi)$ for $0 \leqslant \xi \leqslant t$. Assume that R_1 and b^I are known. Choose the function $g_1{}^I(t)$ in (4.6) such that

$$v^I(a, t) = R_1(t)\, v^{II}(a, t) + b^I(t). \tag{4.13}$$

We have

LEMMA 4.2. *$u(x, t) = v(x, t)$ for $0 \leqslant x \leqslant a$, $t \geqslant 0$ if and only if* (4.13) *holds. Thus* (4.13) *can be considered as the missing boundary condition.*

Proof. Assume that $v(x, t)$ satisfies (4.13). Choose $g_1^{II}(t)$ in (4.9) such that $y^{II}(a, t) = g_1^{II}(t) = v^{II}(a, t)$. Then by (4.12) and (4.13) also $y^I(a, t) = v^I(a, t)$ and by lemma 4.1 we obtain $u(x, t) = v(x, t)$ for $0 \leqslant x \leqslant a$, $t \geqslant 0$. Conversely, if $u(x, t)$ is a solution of (4.1)–(4.3) then it is also a solution of (4.7)–(4.9) and must satisfy (4.12). Therefore, if $u(x, t) = v(x, t)$ for $0 \leqslant x \leqslant a$, $t \geqslant 0$ then $v(x, t)$ must staisfy (4.13). This proves the lemma.

In general $R_1(t)$, $b^I(t)$ are very complicated. There are only some special cases where R_1 and b^I can be represented in a simple way.

1. $r = 0$, i.e. all eigenvalues of $A(a, t)$ are negative, i.e. all characteristics at $x = a$ point out of the region $0 \leqslant x \leqslant a$, $t \geqslant 0$. In this case $y^{II} = y$ and the relation (4.12) is empty. Therefore $v(x, t)$ does not need to satisfy any boundary conditions at $x = a$.

2. $r > 0$ but $A(x, t) = A(a)$ is a constant matrix for $x \geqslant a$. Then the transformation S is independent of x, t and $B \equiv 0$ for $x \geqslant a$. Therefore we can compute the first r components of $y(x, t)$ explicitely. They are given by

$$y^{(j)}(x, t) = f^{(j)}(x + \lambda_j t), \qquad j = 1, 2, \ldots, r;$$

leading to

$$y^{(j)}(a, t) = f^{(j)}(a + \lambda_j t), \qquad j = 1, 2, \ldots, r.$$

The relation (4.12) holds with $R_1 = 0$ and $b^I = (f^{(1)}(a + \lambda_1 t), \ldots, f^{(r)}(a + \lambda_r t))'$.

It is clear that $v(x, t)$ depends very much on the initial values for $x \geqslant a$. There is only one simple case, namely

$$f(x) \equiv f_\infty \equiv \text{const.} \qquad \text{for} \quad x \geqslant a.$$

Then (4.13) becomes

$$v^I(a, t) = f_\infty{}^I \tag{4.14}$$

$v^{\mathrm{I}}(a, t)$ can also be obtained in a fairly simple way if we require only that B has the form

$$B = \begin{pmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{pmatrix}.$$

(4.14) is equivalent with

$$\partial v^{\mathrm{I}}(a, t)/\partial x = 0, \tag{4.14}'$$

because (4.14)′ implies $\partial v^{\mathrm{I}}(a, t)/\partial t = 0$, i.e. $v^{\mathrm{I}}(a, t) = v^{\mathrm{I}}(a, 0) = f_\infty{}^{\mathrm{I}}$. Thus the boundary conditions (3.4) (extrapolation of all the variables at $x = a$) can be used.

Without restriction we can assume that $f_\infty = 0$. Otherwise we would consider the function $\tilde{u} = u - f_\infty$. Then also $b^{\mathrm{I}} = 0$ and the relation (4.13) becomes

$$v^{\mathrm{I}}(a, t) = 0, \qquad \text{for} \quad t \geqslant 0.$$

The last relation means that we set the characteristic variables associated with the "ingoing" characteristics equal to zero, or as B. Engquist and A. Majda [1] call it, that no reflection takes place at the boundary. Therefore this principle is useful if we subtract from the solution its constant state at infinity.

3. $f(x) = 0$ for $x \geqslant a$, $B \neq 0$, but the solutions are highly oscillatory in time, i.e.

$$y^{\mathrm{II}}(a, t) = g^{\mathrm{II}}(a, t) = e^{i\omega t}\phi(t)$$

where $|\omega| \gg 1$ and $\phi(t)$ is a smooth function. Assume, for simplicity, that A and B do not depend on t. Then we can solve (4.7) by Laplace transformation. The transformed equations are

$$s\hat{y} = A\, d\hat{y}/dx + B\hat{y},\ \hat{y}^{\mathrm{II}}(a, s) = \hat{g}^{\mathrm{II}}(a, s),$$

which can be written as

$$d\hat{y}/dx = s(A^{-1} - s^{-1}A^{-1}B)\hat{y},\ \hat{y}^{\mathrm{II}}(a, s) = \hat{g}^{\mathrm{II}}(a, s).$$

We want to solve this equation for $|s| \gg 1$. Then $s^{-1}A^{-1}B$ can be considered as a perturbation of A^{-1} and in first approximation we can neglect B which leads us to the previous case. In general we can obtain the solution as an asymptotic series in s^{-1}. Therefore also R_1 has this form. By using the inverse of the Laplace transform, (4.12) can be written as a relation between time derivatives of the solution.

From now on we shall always make

Assumption 4.1. $f(x) \equiv 0$ for $x \geqslant a$, i.e. $b^{\mathrm{I}}(t) = 0$

Assume that A depends on x, t also for $x \geqslant a$. Then $B \neq 0$ and in general y^{I}, y^{II} are coupled and we cannot determine $R_1(t)$ without making detailed calculations of

$y(x, t)$. Only if $A(x, t)$ converges to a constant matrix A_0 as $x \to \infty$ we can do better. We make

Assumption 4.2. The matrix $B(x, t)$ can be written in the form

$$B(x, t) = \phi(x)\, B_1(x, t) \tag{4.15}$$

where $\phi(x)$ is a scalar function with

$$\int_a^\infty |\, \phi(x)|\, dx \leqslant c \tag{4.16}$$

and

$$\| B \| = \sup_{a \leqslant x < \infty, t > 0} |\, B_1(x, t)| \leqslant 1. \tag{4.17}$$

Here $|\, B_1(x, t)|$ denotes the maximum norm at the point x, t.

Assume for example that A is of the form

$$A(x) = A_0 + \frac{1}{x^2} A_1(x, t), \qquad x \geqslant a$$

and that the eigenvalues of A_0 are all distinct. Then S is of the form

$$S = S_0 + \frac{1}{x^2} S_1(x, t), \qquad \text{i.e.} \quad B = \frac{1}{x^2} B_1 .$$

Thus

$$\phi = \text{const.}/x^2, \qquad \int_a^\infty |\, \phi \,|\, dx \leqslant \text{const.}/a.$$

We need

LEMMA 4.3. *Consider the system*

$$\partial y/\partial t = \Lambda\, \partial y/\partial x + \phi(x) G(x, t)$$

for $x \geqslant a$, $t \geqslant 0$ *with zero initial and boundary conditions*

$$y(x, 0) = 0 \qquad \textit{for} \quad x \geqslant a \; y^{II}(a, t) = 0 \qquad \textit{for} \quad t \geqslant 0.$$

Assume there is a constant $\lambda_0 > 0$ *such that*

$$\inf_{t, x, j} |\, \lambda_j \,| \geqslant \lambda_0 . \tag{4.18}$$

Then

$$\| y \| \leqslant c/\lambda_0 \| G \|, \qquad \| y \| = \sup_{x, t, i} |\, y^{(i)} \,|.$$

Proof. We need to prove the estimate only for scalar equations

$$\partial y/\partial t = \lambda(x, t)\, \partial y/\partial x + \phi(x)G(x, t)$$

with appropriate boundary conditions. Using the method of characteristics we obtain

$$dy/ds = \phi(x)G(x, t),\ dt/ds = 1,\ dx/ds = -\lambda(x, t).$$

Therefore

$$\begin{aligned} |\, y(x(s_0), t(s_0))| &= \left| \int_0^{s_0} \phi(x(s))\, G(x(s), t(s))\, ds \right| \leqslant \| G \| \int_0^{s_0} |\, \phi(x(s))|\, ds \\ &\leqslant \| G \| \int_0^{\infty} |\, \phi(x)|\, |\, \lambda^{-1}(x)|\, dx \leqslant (c/\lambda_0) \| G \|, \end{aligned}$$

which proves the lemma.

The last lemma gives us

THEOREM 4.1. *Assume that the assumptions* 4.1 *and* 4.2 *are valid and that* (4.18) *holds. If* $c/\lambda_0 < 1$ *then the solution of* (4.7) *can be obtained by the iteration process*

$$\begin{gathered} \partial y_{n+1}/\partial t - \Lambda\, \partial y_{n+1}/\partial x = \phi(x)\, B_1(x, t)\, y_n \\ y_{n+1}(x, 0) = 0, \qquad y^{\mathrm{II}}_{n+1}(a, t) = g^{\mathrm{II}}_1(t), \\ n = 0, 1,\ldots; \qquad y_0(x, t) \equiv 0. \end{gathered} \tag{4.19}$$

Proof. Let $\tilde{y}_{n+1} = y_{n+1} - y_n$. By lemma 4.3

$$\| \tilde{y}_{n+1} \| \leqslant (c/\lambda_0) \| \tilde{y}_n \|, \qquad n = 1, 2,\ldots$$

and the convergence follows. This proves the theorem.

In the usual way we obtain the estimate

$$\| y - y_1 \| \leqslant \sum_{\nu=1}^{\infty} \| \tilde{y} \| \leqslant \frac{1}{1 - c/\lambda_0} \| y_2 - y_1 \| = \mathcal{O}((c/\lambda_0))$$

and by (4.19)

$$\begin{gathered} y_1^{\mathrm{I}}(x, t) \equiv 0 \\ \partial y^{\mathrm{II}}_1/\partial t - \Lambda_2\, \partial y^{\mathrm{II}}_1/\partial t = 0, \qquad y^{\mathrm{II}}_1(x, 0) = 0, \qquad y^{\mathrm{II}}_1(a, t) = g^{\mathrm{II}}_1(t). \end{gathered} \tag{4.20}$$

Thus if we allow an error of order $\mathcal{O}(c/\lambda_0)$, extrapolation or the principle of no reflection at $x = a$ is appropriate. If this error is not acceptable then we have to compute y_2^{I} defined by

$$\partial y_2^{\mathrm{I}}/\partial t - \Lambda_1\, \partial y_2^{\mathrm{I}}/\partial x = \phi B_{12} y^{\mathrm{II}}_1, \qquad y_2^{\mathrm{I}}(x, 0) = 0, \tag{4.21}$$

where B_{12} is defined by

$$B_1 = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

Then

$$\| y^{\mathrm{I}} - y_2^{\mathrm{II}} \| = \mathcal{O}((c/\lambda_0)^2),$$

and we assume that this error is tolerable.

If one is interested in the whole time dependent process then not much is gained because to compute $y_2^{\mathrm{I}}(a, t)$ as a function of $g_1^{\mathrm{II}}(t)$ we have to solve (4.20) and (4.21) completely. Thus one should instead make a so large that an error of order $\mathcal{O}(c/\lambda_0)$ can be tolerated.

There is one exception. If $c\| B_{12} \|$ is small, i.e. the in- and outgoing waves are almost decoupled, then y_2^{I} is almost zero and the complete solution is essentially given by (4.20). In this case the principle of no reflection is appropriate.

The situation becomes more favourable if we are only interested in the steady state solution, i.e. consider the case that

$$g_1^{\mathrm{II}}(t) = g_1^{\mathrm{II}}(\infty) \qquad \text{and} \qquad B_1(x, t) \to \tilde{B}_1(x), \qquad \phi(x, t) \to \tilde{A}(x).$$

Then for $t \to \infty$

$$y_1^{\mathrm{I}} \equiv 0, \qquad y_1^{\mathrm{II}} \to g_1^{\mathrm{II}}(\infty), \qquad y_2^{\mathrm{I}}(x) \to \int_{\infty}^{x} \phi \Lambda_1^{-1} B_{12}\, d\xi\, g_1^{\mathrm{II}}(\infty).$$

Thus y_2 satisfies in the limit the relation

$$y_2^{\mathrm{I}}(a, t) = C g_1^{\mathrm{II}}(\infty), \qquad C = \int_{\infty}^{a} \phi \Lambda_1^{-1} B_{12}\, d\xi.$$

Therefore, by lemma 4.2, we should use for $v(x, t)$ at $x = a$ the boundary conditions

$$v^{\mathrm{I}}(a, t) = C v^{\mathrm{II}}(a, t).$$

Observe, however, that one has to know the asymptotic expansion of A in detail to compute C.

5. Problems in Two Space Dimensions

We start with an example. Consider the system

$$\frac{\partial w}{\partial t} = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial w}{\partial x} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial w}{\partial y}, \qquad w = \begin{pmatrix} u \\ v \end{pmatrix}, \tag{5.1}$$

on the domain Ω as given in fig. 1 or fig. 2. For $t = 0$ initial values

$$u(x, y, 0) = v(x, y, 0), \qquad (x, y) \in \Omega, \qquad t = 0, \tag{5.2}$$

and for $(x, y) \in \partial\Omega$ boundary conditions are given. In particular we assume that on B_1

$$v(0, y, t) = h(y), \qquad t \geqslant 0, \qquad x = 0, \tag{5.3}$$

and on B_3, B_4

$$u(x, y, t) = 0, (x, y) \in B_3 \cup B_4, \qquad t \geqslant 0. \tag{5.4}$$

Without restriction we can assume that B_3, B_4 are given by $x \geqslant a$, $y = 1$ and $x \geqslant a$, $y = 0$ respectively.

We want to determine the solution of the above on Ω_1 only. Therefore we need one relation between u, v on B_2. The boundary conditions (5.4) imply that for $x \geqslant a$ the solution of our problem can be expanded in Fourier series

$$u = \sum_{\omega=1}^{\infty} \hat{u}(x, \omega, t) \sin \pi\omega y, \qquad v = \hat{v}_0(x, t) + \sum_{\omega=1}^{\infty} \hat{v}(x, \omega, t) \cos \pi\omega y. \tag{5.5}$$

Introducing (5.5) into (5.1) gives us for $x \geqslant a$

$$\partial\hat{v}_0/\partial t = -\partial\hat{v}_0/\partial x, \tag{5.6}$$

$$\frac{\partial \hat{w}}{\partial t} = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial \hat{w}}{\partial x} + \pi\omega \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \hat{w}, \qquad \omega = 1, 2, \ldots . \tag{5.7}$$

For every fixed frequency $\omega \neq 0$ the system (5.7) is of the same form as the onedimensional problem (4.1) and the results of the last section apply. If we Laplace transform (5.7) with respect to t and we are only interested in solutions with $| s | \gg | \omega |$ then we can use the same kind of asymptotic expansion as before.

If $| s | \ll | \omega |$, i.e. we are interested in quasistationary problems then we can replace (5.6), (5.7) by

$$\partial\hat{v}_0/\partial x = 0 \tag{5.8}$$

$$\begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial \hat{w}}{\partial x} + \pi\omega \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \hat{w} = 0, \qquad \omega = 1, 2, \ldots . \tag{5.9}$$

The solution of (5.9) is given by

$$\hat{u}(x, \omega, t) = -e^{-\omega\pi(x-a)}\hat{v}(a, \omega, t), \hat{v}(x, \omega, t) = e^{-\omega\pi(x-a)}\hat{v}(a, \omega, t),$$

which gives us for $x = a$

$$\hat{u}(a, \omega, t) = -\hat{v}(a, \omega, t). \tag{5.10}$$

Therefore we have determined the desired relation

$$u(a, y, t) = Rv(a, y, t)$$

because if we know v then we can expand it in a cosine series and by (5.10) we obtain the sine series for u. This process can be performed numerically by using the fast Fourier transform.

It is not necessary that the coefficients of (5.1) are constant. We can also consider systems

$$\frac{\partial w}{\partial t} = \begin{pmatrix} \lambda_1(y) & 0 \\ 0 & -\lambda_2(y) \end{pmatrix} \frac{\partial w}{\partial x} + \begin{pmatrix} 0 & a(y) \\ a(y) & 0 \end{pmatrix} \frac{\partial w}{\partial y}$$

where the coefficients depend on y. The steady state solution can again be solved in terms of eigenfunctions

$$\hat{w} = e^{\kappa x}\phi(y), \qquad \text{Real } \kappa < 0,$$

which gives us the desired relation between u and v on B_2. Whether this procedure is feasible numerically depends on how easy it is to compute the eigenfunctions and how many are needed to represent the relation between u and v.

All the results can be carried over to general hyperbolic systems

$$\partial w/\partial t = A_1\, \partial w/\partial x + A_2\, \partial w/\partial y.$$

For example, if we are only interested in the steady state then we can obtain the desired relations on B_2 between the components of w by solving the steady state equations

$$A_1\, \partial w/\partial x + A_2\, \partial w/\partial y = 0$$

in terms of eigenfunction expansions.

We want to point out that there are problems in two space dimensions which can be transformed into one-dimensional form. Consider for example equation (5.1) on some domain Ω containing the origin. We introduce polar coordinates by

$$x = r \cos \theta$$

$$y = r \sin \theta$$

and obtain from (5.1)

$$\frac{\partial w}{\partial t} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \frac{\partial w}{\partial r} + \frac{1}{r} \begin{pmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{pmatrix} \frac{\partial w}{\partial \theta}.$$

Defining new dependent variables by

$$w = \begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ \sin\theta/2 & -\cos\theta/2 \end{pmatrix} u \equiv T(\theta)\, u$$

we get the equivalent system

$$\frac{\partial u}{\partial t} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial u}{\partial r} + \frac{1}{r} T^{-1}(\theta) \begin{pmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{pmatrix} T(\theta) \frac{\partial u}{\partial \theta} + \frac{1}{r} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} u.$$

Therefore, if the initial data are such that $u(r, \theta, 0)$ is independent of θ, then $u(r, \theta, t)$ is independent of θ for all t, $t \geqslant 0$. In particular, we get a nonreflecting boundary condition if $u^{\mathrm{I}} = 0$ is specified at all points on $\partial\Omega$.

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